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Obstruction Calculus for Functors of Artin Rings, I

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In this paper we define and study obstruction theories for morphisms of functors of Artin rings. We prove the existence of a universal obstruction theory, and we give explicit criteria for completeness and for linearity. As applications, we extend several results in the literature, removing the finite-dimensionality of the tangent space and the existence of a vector space of obstructions from the assumptions.

Contents.

An important technical instrument in deformation theory is the study of functors of Artin rings, introduced by Schlessinger in [Sch]. There in particular he introduces the concept of tangent space t_F to a functor.

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Obstruction spaces are a common and useful tool in deformation theory, but there is almost no standard definition of obstructions in the literature. Usually authors either *assume* the existence of a vector space of obstructions, satisfying suitable functoriality conditions (as in [Ar2, Kaw]), or they use an explicit obstruction space for the problem at hand.

The most interesting functors from the viewpoint of deformation theory are those satisfying Schlessinger's conditions (H1) and (H2) (cf. Definition 2.7); following Wahl we will call them functors with good deformation theory, or *Gdt* for short.

The aim of this paper is to propose a definition of obstruction theory for any morphism of functors of Artin rings $\nu\colon F \to G$ in the case that G is *Gdt*; obstruction theory for a functor is an important special case, where the morphism is the projection to the trivial functor $*$. Besides functoriality, we only require that obstructions be a pointed set, so that it makes sense to speak of vanishing obstructions. It is easy to prove the existence of a universal obstruction theory (Theorem 3.2). For *Gdt* functors we prove in Corollary 4.4 that the universal obstruction theory is always complete (that is, liftings exist if and only if obstruction vanishes).

The use of Factorization Theorem 6.2 allows us to extend to *Gdt* functors many results previously proven under the assumption that Schlessinger's condition (H3) (finite-dimensionality of tangent space) also holds, and that there exists a vector space of obstructions. In fact, one of the objectives we had in mind when embarking upon this project was to avoid the use of assumption (H3), which is very often unnatural.

The linearity of obstruction theories is also discussed, and we reach a satisfactory answer (Theorem 6.11) to Artin's question (in [Ar2, p. 169]) as to what extent an obstruction (vector) space is uniquely determined by the functor. We prove that the existence of a vector space as obstruction space for a *Gdt* functor F is equivalent to a Schlessinger-type condition, which we call L (see Definition 2.9); it is also equivalent to the existence of a canonical vector space structure on the universal obstruction theory.

We are therefore led to introduce the concept of functors with *good deformation and obstruction theory* or in short *Gdot* (Definition 2.10). It is easy to see that most functors coming from geometry are *Gdot* (cf. Lemma 2.11), and for them we are able to prove significant generalization of known properties; in particular we generalize to *Gdot* the T^1-lifting theorem (cf. [F-M]) and as a consequence we derive that every *Gdt* group functor is smooth in characteristic zero (for group functors *Gdt* implies *Gdot*).

The paper is a mixture of "classical" and new results; in the first two sections we collect some known definitions and results, both to establish notation and for the convenience of the non-expert reader. There and occasionally in the rest of the paper, some proofs or remarks have been left as exercises.

When we started to write this paper we had in mind specific applications to deformation theory. However, as the number of pages grew, we decided to split the material, collecting here the general theory and leaving for a sequel (in preparation) the applications to deformation theory.

1. NOTATIONAL SET-UP

Let Set_* be the category of pointed sets. We will always denote by $*$ the chosen point of an element V of Set_*, unless V is a vector space when we will assume that the chosen point is zero. The *kernel* of a morphism in Set_* will be the inverse image of the chosen point.

We will work over an arbitrary fixed field k. Let Vsp be the category of k-vector spaces and $Fvsp$ the full subcategory of finite dimensional vector spaces. For a $V \in Vsp$, we denote by $V^\vee$ its k-dual.

Let Art_k be the category of local Artinian k-algebras with residue field k (with as morphisms the local homomorphisms). If $A \in Art_k$, we will denote by $\mathbf{m}_A$ its maximal ideal.

By ϵ and ϵ_i we will always mean indeterminates annihilated by the maximal ideal, and in particular of square zero (e.g., the algebra $k[\epsilon]$ has dimension 2 and $k[\epsilon_1, \epsilon_2]$ has dimension 3 as a k-vector space).

Let $\widehat{Art}_k$ be the category of complete noetherian local k-algebras R such that $R_n = R/\mathbf{m}_R^n$ is in Art_k for all $n \in \mathbb{N}$. Note that Art_k is a subcategory of $\widehat{Art}_k$.

Exercise. Let $R \in \widehat{Art}_k$ and $A \in Art_k$, and choose n such that $\mathbf{m}_A^n = 0$. Then $\operatorname{Hom}(R, A) = \operatorname{Hom}(R_n, A)$ (for the definition of R_n see above).

DEFINITION 1.0. A *small* extension e in Art_k (resp. $\widehat{Art}_k$) is a short exact sequence

$$e\colon 0 \to M \to B \to A \to 0,$$

where $B \to A$ is a morphism in Art_k (resp. $\widehat{Art}_k$), and M is annihilated by the maximal ideal of B (that is, as a B-module it is a k-vector space). In the course of the paper, for every small extension e as above, we shall let $K(e) = M$, $S(e) = B$, $T(e) = A$ (the letters should be a reminder of kernel, source, target). The k-vector space $K(e)$ is called the *kernel* of e. A small extension e will be called *principal* if $\dim_k K(e) = 1$.

For $A \in \widehat{Art}_k$ and $M \in Fvsp$ let $Ex(A, M)$ be the vector space of small extensions of A with kernel M.

LEMMA 1.1. *$Ex(-,-)$ is contravariant in the first variable and covariant in the second.*

Proof. Fix an extension $0 \to M \to B \to A \to 0$ in $Ex(A, M)$. Given a morphism $f\colon M \to N$ in $Fvsp$, define an extension $0 \to N \to B' \to A \to 0$ by letting $B'' = B \oplus N$ (with product $(b,n)(b',n') = (bb', b_0 n' + b'_0 n)$, where $b \to b_0$ is the quotient map $B - k$). Then let B' be the quotient of B'' by the ideal $\{(m, f(m)) | m \in M\}$. Given a morphism $\pi\colon A' \to A$ in Art_k, define an extension $0 \to M \to B' \to A' \to 0$ by letting $B' = A' \times_A B$. ∎

If $f\colon M \to N$ (resp. $\pi\colon B \to A$) is a morphism in $Fvsp$ (resp. $\widehat{Art}_k$), we denote by $f_*\colon Ex(A, M) \to Ex(A, N)$ (resp. $\pi^*\colon Ex(A, M) \to Ex(B, M)$) the induced maps.

Exercise.

(1) Prove that $f_* \pi^* = \pi^* f_*\colon Ex(A, M) \to Ex(B, N)$.

(2) Let $A \in \widehat{Art}_k$. Then there exists a small extension

$$0 \to M \to B \to A \to 0$$

such that the pushforward map $M^\vee \to Ex(A, k)$ is an isomorphism in $Fvsp$.

DEFINITION 1.2. A small extension

$$0 \to M \to B \to A \to 0$$

will be called *trivial* if it corresponds to $0 \in Ex(A, M)$, or equivalently if it splits. In this case we will also write B as $A \oplus M$.

Remark. What we call $Ex(A, M)$ is called $Exal(A, M)$ in [Il] and $T^1(A/k, M)$ in [L-S].

DEFINITION 1.3. We denote by *Smex* the category whose objects are small extensions in Art_k. A morphism of small extensions $\alpha\colon e_1 \to e_2$ is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\alpha_M"] & B_1 \arrow[r] \arrow[d, "\alpha_B"] & A_1 \arrow[r] \arrow[d, "\alpha_A"] & 0 \\
0 \arrow[r] & M_2 \arrow[r] & B_2 \arrow[r] & A_2 \arrow[r] & 0.
\end{tikzcd}$$

Exercise. Prove that $\alpha_{M*}(e_1) = \alpha_A^*(e_2) \in Ex(M_2, A_1)$.

We will also consider the subcategory *Psmex* of *Smex* of *principal* small extensions, that is, those such that the kernel is a one-dimensional vector space. A principal small extension will be called *curvilinear* if, for some

$n \in \mathbb{N}$, it is isomorphic to

$$0 \to k \xrightarrow{\cdot t^n} k[t]/t^{n+1} \to k[t]/t^n \to 0.$$

We shall denote by $\epsilon \in Ex(k, k)$ the (trivial) extension

$$0 \to k \xrightarrow{\epsilon} k[\epsilon] \to k \to 0.$$

2. FUNCTORS OF ARTIN RINGS AND SCHLESSINGER CONDITIONS

We now introduce the functors we will study in this paper.

DEFINITION 2.1. A *functor of Artin rings* (or sometimes just *functor*, when no confusion is likely to arise) is a covariant functor $F\colon Art_k \to Set_*$ such that $F(k) = *$; such functors together with natural transformations form a category, which we denote by *Fun*.

Remark. In fact, it is equivalent to ask that F be a functor with values in *Set*; the chosen point is then determined by the requirement that $F(k)$ be one point together with the fact that k is an initial object in the category Art_k.

EXAMPLE 2.2. The constant functor $*$ defined by $*(A) = \{*\}$ for all $A \in Art_k$ is both an initial and a final object in *Fun*; we sometimes call it the trivial functor.

EXAMPLE 2.3. Let R be a local k-algebra with residue field k. Let $h_R \in Fun$ be the functor $h_R(A) = \mathrm{Hom}(R, A)$. The map $R \to h_R$ is a contravariant functor from local k-algebras to *Fun*; in fact, it embeds the dual category of $\widehat{Art}_k$ as a full subcategory of *Fun* (see [Sch]).

Exercise. (i) For every $A \in Art_k$ and $F \in Fun$ there exists an obvious bijection between $F(A)$ and $\mathrm{Mor}_{Fun}(h_A, F)$.

(ii) More generally, for every $R \in \widehat{Art}_k$, there is a bijection between $\mathrm{Mor}_{Fun}(h_R, F)$ and the inverse limit of the sets $F(R/\mathbf{m}_R^n)$.

(iii) If $A, B \in Art_k$, then $h_{A\otimes B} = h_A \times h_B$. More generally, if $R, S \in \widehat{Art}_k$, then $h_{R \hat{\otimes} S} = h_R \times h_S$, where $R \hat{\otimes} S$ is defined as the inverse limit of $R_n \otimes S_n$.

EXAMPLE 2.4. Let V be a k-vector space. Define $T_V \in Fun$ by $T_V(A) = V \otimes \mathbf{m}_A$. Then $T_V \in Fun$. Note that, if $V = W^\vee$, then $T_V = h_R$, where R is the symmetric algebra of W.

DEFINITION 2.5. Given a morphism $\nu: F \to G$ in *Fun*, we define the *tangent space* to F to be the set $t_F = F(k[\epsilon])$; we define the *relative tangent space* t_ν to ν to be the kernel of $t_F \to t_G$.

DEFINITION 2.6. Given a functor $F \in Fun$ and morphisms $A' \to A$, $A'' \to A$ in Art_k, let $(*)$ be the natural map

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A''). \qquad (*)$$

The functor F is called *left-exact* if $(*)$ is bijective; F is called *homogeneous* if $(*)$ is bijective whenever $A'' \to A$ is surjective (cf. [Se]). It is immediate to observe that the functors h_R are left-exact.

DEFINITION 2.7. Let F be a functor in *Fun*. The following are called *Schlessinger conditions*:

(H1) Map $(*)$ is surjective if $A'' \to A$ is a principal small extension.

(H2) Map $(*)$ is bijective if $A'' = k[\epsilon]$, $A = k$.

(H3) Conditions (H1) and (H2) hold and $\dim_k F(k[\epsilon])$ is finite (cf. Lemma 2.12).

(H4) Map $(*)$ is bijective if $A'' \to A$ is a principal small extension.

Remark. Condition (H1) (resp. (H4)) is equivalent to requiring that $(*)$ be surjective (resp. bijective) whenever $A'' \to A$ is a small extension; (H2) is equivalent to requiring that $(*)$ be bijective whenever $A'' \to A$ is a small extension, $A = k$. When we need it, we will use these alternative formulations of the conditions without further notice.

It is clear that *left-exact* $\Rightarrow$ *homogeneous* $\Rightarrow$ (H1), (H2) *and* (H4); we shall prove in Corollary 6.3 that F satisfies conditions (H1), (H2), and (H4) if and only if F is left-exact.

DEFINITION 2.8. If a functor F satisfies (H1), (H2) then F is called a *functor with good deformation theory* (cf. [Wa, p. 532]); such functors form a subcategory *Gdt* of *Fun*.

Remark. All functors in the examples above are in *Gdt*. For an example of a functor in *Fun* but not in *Gdt*, see Example 2.13.

Notation. When we know that $(*)$ is bijective, given $(a, b) \in F(A') \times_{F(A)} F(A'')$, we will denote by $a \oplus b$ its inverse image in $F(A' \times_A A'')$.

DEFINITION 2.9. Let F be a functor in *Fun*. F satisfies condition (L) (which is related to linearity of obstructions, see Theorem 6.12) if the following holds:

(L) For every small extension

$$0 \to M \to B \to A \to 0,$$

let $C = B \times_k B/I$ where $I = \{(b, b) | b \in M\}$ and let p and q be the natural maps

$$F(C) \xrightarrow{p} F(A \times_k A) \xrightarrow{q} F(A) \times F(A);$$

then $q^{-1}(\Delta_{F(A)}) \subset p(F(C))$, where $\Delta_{F(A)}$ is the diagonal. That is, every element of $F(A \times_k A)$ having equal projections in $F(A)$ must lift to $F(C)$.

DEFINITION 2.10. If a functor F satisfies (H1), (H2), and (L) then F is called a *functor with good deformation and obstruction theory*. Such functors form a subcategory *Gdot* of *Gdt*. In most concrete cases (e.g., deformations of schemes, cf. [Sch, p. 220]) one can verify condition (L) by using the following lemma.

LEMMA 2.11. *In the notation of Definition* 2.6 *assume* $(*)$ *bijective when* $A = k$. *Then* F *satisfies conditions* (H2) *and* (L).

Proof. By assumption the map q is bijective. Let $\delta\colon A \to A \times_k A$ be the diagonal map; then $q^{-1}\Delta_{F(A)} = \delta(F(A))$. It is enough to show that $p(F(C))$ contains $\delta(F(A))$, which is immediate as δ factors through C. ∎

LEMMA 2.12 (Schlessinger). *Let* F *be a Gdt functor*.

(1) t_F *has a natural structure of* k-*vector space* (*hence* (H3) *makes sense*).

(2) *Let* $V \in Fvsp$. *Then* $F(k \oplus V)$ *is canonically in bijection with* $t_F \otimes V$.

(3) *Given a small extension*

$$0 \to M \to B \to A \to 0$$

there is a canonical transitive action of $t_F \otimes M$ *on each fiber of* $F(B) \to F(A)$, *compatible in the obvious sense with morphisms of small extensions*.

Proof. (1) Let $v_1, v_2 \in t_F$. Consider v_i as an element of $F(k[\epsilon_i])$, and let $v = v_1 \oplus v_2 \in F(k[\epsilon_1, \epsilon_2])$. Let $\lambda_1, \lambda_2 \in k$, and define $\lambda_1 v_1 + \lambda_2 v_2$ to be $\phi(v_1 \oplus v_2)$, where $\phi\colon k[\epsilon_1, \epsilon_2] \to k[\epsilon]$ is given by $\phi(\epsilon_i) = \lambda_i \epsilon$. It is easy to check that this makes t_F into a k-vector space.

(2) Induction on $\dim V$. If $\dim V = 1$, there is nothing to prove. Otherwise, write $V = U \oplus W$; then $F(k \oplus V) = F(k \oplus U) \times F(k \oplus W)$ by axiom (H2), and the result follows.

(3) If the extension is split, that is, if $B = A \times_k (k \oplus M)$, by (H2) we have $F(B) = F(A) \times (t_F \otimes M)$. Otherwise, let $C = B \times_A B$. Then $C = B \oplus M$, hence $F(C) = F(B) \times (t_F \otimes M)$; on the other hand, by (H1), $F(C)$ surjects into $F(B) \times_{F(A)} F(B)$. hence we get a map $F(B) \times (t_F \otimes M) \to F(B)$; it is easy to check that it is an action and has the claimed properties. ∎

Exercise. (i) If $\nu\colon F \to G$ is a morphism in *Gdt*, the induced map $t_F \to t_G$ is linear.

(ii) Let $F \in Gdt$. Then F is trivial if and only if $t_F = 0$.

(iii) Let $F, G \in Gdt$. Then $F \times G$ is in *Gdt* and $t_{F\times G} = t_F \oplus t_G$.

Usually all functors of Artin rings arising from deformation theory problems have a good deformation theory, and in order to obtain fine results we will, in a later section of the paper, restrict our attention to *Gdt* functors. However, non-*Gdt* functors can be constructed very easily in terms of *Gdt* ones.

EXAMPLE 2.13. Let $\nu\colon F \to G$ be a morphism in *Gdt*. Let $H = \nu(F)$ be the image (defined in the obvious way). Then H is an element of *Fun*, but in general not of *Gdt*. Consider for instance the case where $F = G = h_R$ with $R = k[[t]]$, and let $\nu\colon F \to G$ be induced by $t \to t^2$. Then $\nu(F)$ is nontrivial, but $\nu(F)(k[\epsilon]) = *$.

DEFINITION 2.14. For any morphism $\nu\colon F \to G$ in *Fun*, define a covariant functor $\tilde{\nu}\colon Smex \to Set_*$ by setting for every small extension

$$e\colon 0 \to M \to B \to A \to 0$$
$$\tilde{\nu}(e) = G(B) \times_{G(A)} F(A).$$

Remark. $\tilde{\nu}(\epsilon) = t_G$, independently of ν and F. If $G = *$, then $\tilde{\nu}(e) = F(A)$.

DEFINITION 2.15. A morphism $\nu\colon F \to G$ in *Fun* is *smooth* if, for every $e \in Smex$,

$$e\colon 0 \to M \to B \to A \to 0$$

the natural map $F(B) \to \tilde{\nu}(e)$ is surjective. For a justification of this name as well as the main properties of smooth morphisms we refer the reader to [Sch]. Here we just recall that, if ν is smooth, then $F(A) \to G(A)$ is surjective for all $A \in Art_k$. A functor $F \in Fun$ is smooth if the morphism $F \to *$ is. This is equivalent to requiring that $F(B) \to F(A)$ be surjective for every small extension $B \to A$.

DEFINITION 2.16 (cf. [Sch]). Let F be a functor in *Fun* and R an algebra in $\widehat{Art_k}$. R is said to *prorepresent* F if we are given an isomorphism $h_R \to F$. Such an F is then called *prorepresentable*; $h_R \to F$ is determined by F up to canonical isomorphism. R is said to be a *hull* for F if we are given a morphism $h_R \to F$ which is smooth and bijective on tangent spaces. A hull, if it exists, is unique up to non-canonical isomorphism.

3. DEFINITION OF RELATIVE OBSTRUCTION THEORY

In this section we develop relative obstruction theories for a morphism $\nu: F \to G$ in *Fun* under "minimal" assumptions, that is, we require only that G be in *Gdt*. In fact Definition 3.1 and Theorem 3.2 hold in complete generality; however, without assumptions on G we cannot prove Proposition 3.3 (more precisely the key Lemma 3.4), and therefore the more general notion seems to be of little interest.

DEFINITION 3.1. Let $\nu: F \to G$ be a morphism in *Fun*, and assume $G \in Gdt$. A *relative obstruction theory* (V, v_e) for ν is the data of an *obstruction space* $V \in Set_*$ and, for every small extension $e \in Smex$, of an *obstruction map* $v_e: \tilde{\nu}(e) \times K(e)^\vee \to V$, where $K(e)$ is the kernel of e, and $\tilde{\nu}(e)$ was defined in Definition 2.14. The maps v_e must satisfy the following two conditions:

(1) $v_e(0, 1) = *$, where $1 \in k^\vee$ is the identity.

(2) (Base change) For every morphism $e' \to e$ in *Smex*, the diagram

$$\begin{array}{ccc} \tilde{\nu}(e') \times K(e)^\vee & \longrightarrow & \tilde{\nu}(e) \times K(e)^\vee \\ \big\downarrow & & \big\downarrow{\scriptstyle v_e} \\ \tilde{\nu}(e') \times K(e')^\vee & \xrightarrow{v'_e} & V \end{array}$$

commutes.

A *morphism* $(V, v_e) \to (V', v'_e)$ of relative obstruction theories is a morphism $\alpha: V \to V'$ such that $v'_e = \alpha \circ v_e$. A relative obstruction theory (O_v, ob_e) is *universal* if for every relative obstruction theory (V, v_e) there exists a unique morphism $(O_\nu, ob_e) \to (V, v_e)$. A relative obstruction theory (V, v_e) is called *trivial* if $V = \{*\}$. If e is a small extension with kernel k, we will often write $v_e(\cdot)$ instead of $v_e(\cdot, 1)$.

If $F \in Fun$, we call *obstruction theory* for F a relative obstruction theory for the morphism $F \to *$; the universal obstruction theory is then denoted O_F.

Exercises. Let ν be a morphism in *Fun* with *Gdt* target, and let (V, v) be any obstruction theory for ν. Let e be any small extension.

(i) $v_\epsilon(0, f) = *$ for every $f \in k^\vee$ (use $f_* \epsilon = \epsilon$).

(ii) If $K(e) = 0$ then $v_e = *$ (use π^* where $\pi: A \to k$ is the canonical projection).

(iii) For every $x \in \tilde{\nu}(e)$, $v_e(x, 0) = *$ (use (ii) and base change).

THEOREM 3.2. *In the above notation there exists a unique universal relative obstruction theory* (O_ν, ob_e).

Proof. The unicity is clear. To prove existence, let O_ν be the quotient of the set

$$\hat{O} = \bigcup_{e \in Smex} \tilde{\nu}(e) \times K(e)^\vee$$

by the (base change) equivalence relation $\sim$ generated by

$$\tilde{\nu}(e') \times K(e)^\vee \to \left(\tilde{\nu}(e) \times K(e)^\vee\right) \times \left(\tilde{\nu}(e') \times K(e')^\vee\right),$$

for all morphisms $e' \to e$ in *Smex*. Take the equivalence class of $(0,1) \in \tilde{\nu}(\epsilon) \times K(\epsilon)^\vee$ as the distinguished point of O_ν. ∎

Remark. For later use we give another (essentially equivalent) description of the universal obstruction theory. We identify the set

$$\tilde{O} = \{(e,a) | e \in Ex(A,k), A \in Art, a \in \tilde{\nu}(e)\}$$

with a subset of $\hat{O}$ by mapping $(e,a) \to (e,a,1)$; there is also a retraction r: $\hat{O} \to \tilde{O}$ given by $(e,a,\phi) \to (\phi_* e, \nu\phi(a))$, and clearly $x \sim r(x)$ for every $x \in \hat{O}$.

Therefore if we call $\tilde{\sim}$ the restriction of $\sim$ to $\tilde{O}$ we have a canonical isomorphism $\tilde{O}/\tilde{\sim} = O_\nu$.

Note that this implies that an obstruction theory is determined once one knows V and $v_e(a)$, for every $A \in Art_k$, every $e \in Ex(A,k)$, and every $a \in \tilde{\nu}(e)$.

Remark. The universality of O_ν implies its functoriality; more precisely for every commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\alpha} & H \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \mu} \\ G & \xrightarrow{\beta} & L \end{array}$$

there exists a natural morphism of pointed sets $O(\alpha,\beta)$: $O_\nu \to O_\mu$. In order to avoid heavy notation we shall frequently write, when confusion is not possible, (α,β) instead of $O(\alpha,\beta)$, similarly if ν: $F \to G$ is a morphism of functors of Artin rings we use both ν and $O(\nu)$ to denote the induced obstruction map $O_F \to O_G$.

Exercise. Let $F, G \in Fun$. Prove that $O_{F \times G} = O_F \times O_G$.

The name obstruction theory is motivated by the following proposition.

PROPOSITION 3.3. *Let $\nu: F \to G$ be a morphism in Fun, with $G \in Gdt$; let (V, v_e) be a relative obstruction theory for ν. Let e be the small extension*

$$0 \to M \to B \xrightarrow{\pi} A \to 0$$

and let $(\bar{b}, a) \in \tilde{\nu}(e) = G(B) \times_{G(A)} F(A)$. If $(\bar{b}, a)$ is contained in the image of $F(B)$, then $v_e((\bar{b}, a), f) = *$ for every $f \in M^\vee$.

Proof. (1) We can assume $M = k$ by replacing e with $f_*(e) \in Ex(A, k)$.

(2) We can assume that e is trivial, by replacing e by $\pi^* e$ and $(\bar{b}, a)$ by $(G(\delta)(\bar{b}), b)$, where $\delta: B \to B \times_A B$ is the diagonal homomorphism and $b \in F(B)$ maps to $(\bar{b}, a)$.

(3) Since e is trivial we can write $e = \psi^* \epsilon$ where $\psi: A \to k$ is the canonical projection, so we can assume $e = \epsilon$ and $a = *$.

(4) By Lemma 3.4 we have $v_e((\bar{b}, *), f) = v_\epsilon(\nu(b), f) = *$. ∎

LEMMA 3.4. *In the same assumptions as Proposition 3.3, let $Z \subset t_G$ be the linear subspace generated by the image of t_F. Then for every $a \in t_G$, $b \in Z$ we have $v_\epsilon(a + b, 1) = v_\epsilon(a, 1)$; in particular Z is contained in the kernel of the obstruction map $v_e(-, 1): t_G \to V$.*

Proof. It is sufficient to prove that for every $a \in t_G$, $b \in \nu(t_F)$, and $\alpha \in k$ we have $v_\epsilon(a + \alpha b) = v_\epsilon(a)$.

Let $c \in t_F$ such that $\nu(c) = b$ and consider the following commutative diagrams (recall that ϵ, ϵ_1, and ϵ_2 are indeterminates annihilated by the maximal ideal and in particular with square zero),

$$\begin{tikzcd} 0 \arrow[r] & k \arrow[r, "\epsilon"] \arrow[d, equal] & k[\epsilon, \epsilon_1] \arrow[r, "\pi"] \arrow[d, "\phi"] & k[\epsilon_1] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & k \arrow[r, "\epsilon"] & k[\epsilon] \arrow[r] & k \arrow[r] & 0 \end{tikzcd}$$

where $\phi(\epsilon) = \epsilon, \phi(\epsilon_1) = 0$;

$$\begin{tikzcd} 0 \arrow[r] & k \arrow[r, "\epsilon"] \arrow[d, equal] & k[\epsilon, \epsilon_1] \arrow[r, "\pi"] \arrow[d, "\psi"] & k[\epsilon_1] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & k \arrow[r, "\epsilon_2"] & k[\epsilon_2] \arrow[r] & k \arrow[r] & 0 \end{tikzcd}$$

where $\psi(\epsilon) = \epsilon_2, \psi(\epsilon_1) = \alpha\epsilon_2$. Since $G \in Gdt$ there exists $a \oplus b \in G(k[\epsilon, \epsilon_1])$ such that $\phi(a \oplus b) = a, \pi(a \oplus b) = b, \psi(a \oplus b) = a + \alpha b$.

The result follows by the base change property applied to the above commutative diagrams and the element $(a \oplus b, c) \in G(k[\epsilon, \epsilon_1]) \times_{G(k[\epsilon_1])} F(k[\epsilon_1])$. ∎

DEFINITION 3.5. Let $\nu\colon F \to G$ be a morphism in *Fun*, such that G is *Gdt*. An element in O_ν is said to be a *curvilinear obstruction* if it is in the image of ob_e for some curvilinear extension e.

LEMMA 3.6. *Let F be a Gdt functor. Let $x_1, x_2 \in O_F$. Then there exists a small extension e, an $a \in F(T(e))$, and $f_1, f_2 \in K(e)^\vee$ such that $x_i = ob_e(a, f_i)$.*

Proof. We can assume that $x_i = ob_{e_i}(a_i)$ for some small extension

$$e_i\colon 0 \to k \to B_i \to A_i \to 0$$

and for some $a_i \in F(A_i)$. Let $A = A_1 \times_k A_2$, $B = B_1 \times_k B_2$, and let e be the extension induced by the surjection $B \to A$. By (H1) we can lift (a_1, a_2) to an $a \in F(A)$. Choose f_1 and f_2 in $K(e)^\vee$ such that the diagrams

$$\begin{tikzcd} e\colon & 0 \arrow[r] & k \oplus k \arrow[r] \arrow[d, "f_i"] & B \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\ e_i\colon & 0 \arrow[r] & k \arrow[r] & B_i \arrow[r] & A_i \arrow[r] & 0 \end{tikzcd}$$

commute for $i = 1, 2$. Then the statement follows immediately by base change. ∎

A similar result could be proven for ν a morphism in *Gdt*: we will not need this generalization.

4. COMPLETENESS AND LINEARITY

In this section we introduce and study two important properties of obstruction theories.

DEFINITION 4.1. An obstruction theory (V, v_e) is called *complete* if the converse of Proposition 3.3 holds. That is, we require that, for any small extension

$$e\colon 0 \to M \to B \to A \to 0,$$

an element x in $\tilde{\nu}(e)$ lifts to $F(B)$ if and only if, for every $f \in M^\vee$, $v_e(x, f) = *$.

Remark. A morphism $\nu\colon F \to G$ in *Fun* with $G \in Gdt$ is smooth if and only if the trivial obstruction theory is complete (and therefore universal).

Exercise. If ν has a complete obstruction theory then also the universal obstruction theory (O_ν, ob_e) is complete.

The completeness of the universal obstruction theory is a quite restrictive condition on the morphism $\nu\colon F \to G$.

PROPOSITION 4.2. *For a morphism $\nu\colon F \to G$ as above the following are equivalent*:

(1) *O_ν is complete.*

(2) *For every $e \in Ex(A, k), \pi\colon B \to A$ and $a \in \tilde{\nu}(\pi^* e)$, a lifts to $F(S(\pi^* e))$ if and only if $\nu(a)$ lifts to $F(S(e))$.*

(3) *For every $e \in Ex(A, k)$, $a \in \tilde{\nu}(e)$, a lifts to $F(S(e))$ if and only if $ob_e(a, 1) = *$.*

Proof. (1) ⇒ (2). This follows immediately by base change.

(2) ⇒ (3). According to the alternative description of O_ν, $ob_e(a, 1) = *$ if and only if $(e, a) \approx (\epsilon, 0)$ and then (3) is equivalent to saying that, for $(e, a) \approx (e', a')$, a lifts if and only if a' lifts; this can be checked on the generators of the relation $\approx$.

(3) ⇒ (1). Let e be the small extension

$$0 \to M \to B \xrightarrow{\pi} A \to 0$$

and assume $a \in \tilde{\nu}(e)$ satisfies $ob_e(a, f) = *$ for every $f \in M^\vee$. We prove that a lifts to $F(B)$ by induction on $\dim_k M$. If $\dim M = 1$ then a lifts by (3). Assume $\dim M > 1$ and let $f \in M^\vee$ with proper kernel $N \subset M$. Consider the following small extensions and morphisms:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & B \arrow[r, "\pi'"] \arrow[d, equal] & A' \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & B \arrow[r, "\pi"] \arrow[d, "\pi'"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & k \arrow[r] & A' \arrow[r, "\delta"] & A \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is $f_*(e)$; call e' the top row. By (3) applied to $f_*(e)$, a lifts to $a' \in \tilde{\nu}(e')$; by base change, since $N \to M$ is injective, $ob_{e'}(a', g) = *$ for every $g \in N^\vee$. Hence by the inductive hypothesis a' lifts to $F(B)$. ∎

In the absolute case we can simplify the criterion for the completeness of the universal obstruction theory. Let $F \in Fun$: for every small extension e

$$0 \to M \to B \xrightarrow{\pi} A \to 0$$

the universal obstruction map is ob_e: $F(A) \times M^\vee \to O_F$ and, if O_F is complete, $a \in F(A)$ lifts to $F(B)$ if and only if $ob_e(a, f) = *$ for every $f \in M^\vee$.

For $A \in Art_k$ and $a \in F(A)$ define

$$Z(A, a) = \{e \in Ex(A, k) | a \text{ lifts to } F(S(e))\}.$$

PROPOSITION 4.3. *For every functor of Artin rings F the following conditions are equivalent*:

(1) O_F *is complete*.

(2) *For every morphism* π: $B \to A$ *and every small extension* ϕ: $C \to A$ *the map*

$$F(C \times_A B) \to \pi^{-1}\phi(F(C))$$

is surjective. *In particular* (2) *holds if F satisfies* (H1).

(3) *For every* π: $B \to A$ *and* $b \in F(B)$, *the set* $Z(B, b)$ *is a vector subspace of* $Ex(B, k)$ *and* $\pi^* Z(A, \pi(b)) \subset Z(B, b)$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) is exactly Proposition 4.2 applied in the absolute case.

Assume O_F complete and let $e_1, e_2 \in Z(B, b)$, with e_i being the sequence

$$0 \to k \to B_i \xrightarrow{\pi_i} B \to 0.$$

Then by (2), b lifts to $B_1 \times_B B_2$, therefore every linear combination of e_1, e_2 belongs to $Z(B, b)$. The inclusion $\pi^* Z(A, \pi(b)) \subset Z(B, b)$ is exactly (2).

Conversely (3) implies (1) by Proposition 4.2. ∎

COROLLARY 4.4. *If F is a Gdt functor, then* O_F *is complete*.

Proof. Evident. ∎

Given morphisms in *Fun*

$$F \xrightarrow{\nu} G \xrightarrow{\mu} H$$

with H and G in *Gdt*, consider the induced sequence of maps

$$0 \longrightarrow t_\nu \longrightarrow t_{\mu\nu} \xrightarrow{\nu} t_\mu \xrightarrow{ob_\epsilon} O_\nu \xrightarrow{(1, \mu)} O_{\mu\nu} \xrightarrow{(\nu, 1)} O_\mu; \qquad (4.5)$$

by Proposition 3.3, the sequence (4.5) is a complex of pointed sets.

THEOREM 4.6. *If $O_\nu, O_{\mu\nu}, O_\mu$ are complete then* (4.5) *is an exact sequence of pointed sets.*

Proof. Exactness in $O_{\mu\nu}$ and in t_μ is easy and left to the reader. We check the exactness in O_ν; we will use the alternative description of the universal obstruction theory. Let e be the small extension

$$0 \to k \to B \overset{\pi}{\to} A \to 0$$

and let $x = ob_e((b,a),1) \in O_\nu$ such that $(1,\mu)x = ob_e((\mu(b),a),1) = *$; we must prove that $x = ob_\epsilon(t,1)$ for some $t \in t_G$ with $\mu(t) = 0$.

Since $O_{\mu\nu}$ is complete there exists $c \in F(B)$ such that $\pi(c) = a$, $\mu\nu(c) = \mu(b)$. We can assume without loss of generality that $e \in Ex(A,k)$ is the trivial extension, in fact otherwise we can apply the base change property to the diagram

$$\begin{array}{ccc} B \times_A B & \xrightarrow{\pi_2} & B \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow \\ B & \longrightarrow & A \end{array}$$

and get $x = ob((\beta,c),1)$ where $\beta \in G(B \times_A B)$ is such that $\pi_1(\beta) = b$, $\pi_2(\beta) = \nu(c)$. Thus $e = \pi^*\epsilon$ where $\pi\colon A \to k$ is the natural projection; by base change we can assume that $x = ob_\epsilon(t,1)$ with $t \in t_G$. Let t' be the image of t in t_H; as x maps to $*$ in O_μ and O_μ is complete, t' lifts to $s \in t_F$. Then by Lemma 3.4, $x = ob_\epsilon(t - \nu(s), 1)$, and $t - \nu(s) \in t_\mu$. ∎

This theorem can be viewed as a formal justification of the well-known philosophy in deformation theory that, given a "natural" morphism of deformation functors, the induced maps on tangent and obstruction spaces are connected by such an exact sequence. For concrete examples and applications, see for instance [Ran1]; longer exact sequences exist for deformation functors (as in [Fl, Satz 3.4]).

DEFINITION 4.7. Let $\nu\colon F \to G$ be a morphism in *Fun*, with $G \in Gdt$. An obstruction theory (V, v_e) for ν is called *linear* if V is a k-vector space and for every small extension

$$e\colon 0 \to M \to B \overset{\pi}{\to} A \to 0$$

and every $(b,a) \in G(B) \times_{G(A)} F(A)$ the map

$$\theta = ob_e((b,a),-)\colon M^\vee \to V$$

is linear; this is equivalent to saying that v_e is induced by a map $\tilde{\nu}(e) \to V \otimes M$. When V is linear, given an element $x \in \tilde{\nu}(e)$ we will often speak of its image in $V \otimes M$ as *the* obstruction of x. A complete linear obstruction theory for a functor F (that is, for the morphism $F \to *$) is an obstruction space in the usual sense, for instance as in [Kaw].

The trivial obstruction is clearly linear, as well as most obstructions "coming from geometry." A weak form of linearity holds for all complete obstructions.

LEMMA 4.8. *Let e be the small extension*

$$0 \to M \to B \xrightarrow{\pi} A \to 0$$

and let $(b, a) \in G(B) \times_{G(A)} F(A)$ *be a fixed element.*

The map $\theta = ob_e((b, a), -1)$: $M^\vee \to O_\nu$ *has the following properties*:

(1) $\theta(0) = *$.

(2) *If* O_ν *is complete and* $\theta(f) = *$ *then* $\theta(g + \alpha f) = \theta(g)$ *for every* $g \in M^\vee$ *and* $\alpha \in k$.

Proof. (1) This is an easy consequence of Lemma 3.4 and base change.

(2) By base change we can assume that $M = k^2$ and f, g are respectively the projections onto the first and second factor.

By base change applied to

$$\begin{tikzcd} 0 \arrow[r] & k \oplus k \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "\tilde{f}"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & k \arrow[r] & B_1 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

we get $ob_{f_* e}((\tilde{f}(b), a), 1) = *$; since O_ν is complete there exists a lifting $b_1 \in F(B_1)$ of $(\tilde{f}(b), a)$.

Consider now the morphisms of small extensions

$$\begin{tikzcd} 0 \arrow[r] & k \arrow[r] \arrow[d, "{(0,1)}"] & B \arrow[r, "f"] \arrow[d, equal] & B_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & k \oplus k \arrow[r] \arrow[d, "g+\alpha f"] & B \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & k \arrow[r] & B_\alpha \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

Since the composition of $(0, 1)$ and $g + \alpha f$ is independent from α, we have $\theta(g + \alpha f) = ob_{e_1}((b, b_1), 1)$, where e_1 is the upper extension in the above diagram. ∎

For $A \in Art_k$, let δ: $A \to A \times_k A$ be the diagonal map, π_1, π_2: $A \times_k A \to A$ the projections. For any $M \in Fvsp$, let ∇: $Ex(A, M) \to Ex(A \times_k A, M)$ denote $\pi_1^* - \pi_2^*$.

LEMMA 4.9. *Let $F \in Fun$ and $A \in Art_k$, let $a \in F(A \times_k A)$, and $a_i = \pi_i(a) \in F(A)$; fix $e \in Ex(A, k)$. Assume that O_F is complete.*

(1) *If $ob_{\nabla(e)}(a) = *$, then $ob_e(a_1) = ob_e(a_2)$.*

(2) *Assume F admits a complete, linear obstruction theory (V, v_e); then $v_e(a_1) = v_e(a_2)$ implies $v_{\nabla(e)}(a) = *$, and in particular F satisfies condition* (L) (*see Definition* 2.9).

Proof.

(1) Let e be

$$\epsilon: 0 \to k \to B \to A \to 0,$$

hence $e \times e, \nabla(e)$ are respectively

$$0 \to k^2 \to B \times_k B \to A \times_k A \to 0,$$
$$0 \to k \to C \to A \times_k A \to 0,$$

where $C = B \times B/\{(b, b)|b \in K(e)\}$, as in the definition of condition (L). Let p_i: $k^2 \to k$ be the projections. Then $\nabla(e) = p_{1_*} - p_{2_*}(e \times e)$; let $f = p_1 - p_2 \in (k^2)^\vee$. By base change, $ob_{e \times e}(a, f) = *$, hence by Lemma 4.8(2) (which applies because O_F is complete) $ob_{e \times e}(a, p_1) = ob_{e\times e}(a, p_2)$, and again by base change $ob_e(a_1) = ob_e(a_2)$.

(2) By base change, $v_{e\times e}(a, p_1) = v_{e\times e}(a, p_2)$. Applying linearity, $v_{e\times e}(a, f) = 0$, hence by base change again we are done. Condition (L) means that if $a_1 = a_2$ then a lifts to $F(C)$; this follows because (V, v_e) is complete. ▮

We now want to give some criteria for the completeness of the relative obstruction theory to a morphism in *Gdt*. To this end, we introduce the following

Set-up 4.10. Assume we are given a morphism ν: $F \to G$ in *Gdt*. Let

$$\begin{array}{ccc} C \times_A B & \xrightarrow{\gamma} & B \\ \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\delta} \\ C & \xrightarrow{\alpha} & A \end{array}$$

be a diagram in Art_k, with $\alpha \in Psmex$; let $E = C \times_A B$, and let $a \in G(E)$, $b \in F(B)$ and $c \in F(C)$ be elements satisfying $\nu(b) = \gamma(a)$, $\nu(c) = \beta(a)$, $\alpha(c) = \delta(b)$. We say that $d \in F(E)$ is a *required element* if $\nu(d) = a$, $\gamma(d) = b$. Note that we don't ask for $\beta(d) = c$. Note also that by Proposition 3.3 and base change, we have $ob_\gamma((a, b)) = *$. Proposition 4.2 says that O_ν is complete if and only if, for any Set-up 4.10, a required element exists.

PROPOSITION 4.11. *Let $\nu\colon F \to G$ be a morphism in Gdt; let $\theta\colon t_G/t_F \to O_\nu$ be the morphism induced by $t_G \to O_\nu$ according to Lemma* 3.4. *Then O_ν is complete if and only if* $\ker\theta = *$.

Proof. If O_ν is complete then θ is injective by Theorem 4.6. Conversely, assume θ is injective and that we are as in Set-up 4.10. As $F \in Gdt$, there exists $d \in F(E)$ such that $\beta(d) = c$, $\gamma(d) = b$; in particular, letting $a' = \nu(d)$, one has $\gamma(a) = \gamma(a')$, hence there exists a $t \in t_G$ such that $t \cdot a' = a$ (where $\cdot$ is the action of t_G as in Lemma 2.12(3)). If $t = \nu(s)$ with $s \in t_F$, then $t \cdot d$ is a required element and we are done. Therefore we need to prove that $ob_\epsilon(t) = *$.

Let v be a generator of the kernel of γ; there exists a morphism of small extensions $e_1 \to e_2$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & k & \xrightarrow{\cdot\epsilon} & E[\epsilon] & \xrightarrow{\epsilon\mapsto 0} & E & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \epsilon\mapsto v} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & k & \xrightarrow{\cdot v} & E & \xrightarrow{\gamma} & B & \longrightarrow & 0.
\end{array}$$

By base change $ob_{e_1}(a' \oplus t, d) = ob_{e_2}(a, b) = *$ and by base change applied to the morphism $e_1 \to \epsilon$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & k & \xrightarrow{\cdot\epsilon} & E[\epsilon] & \xrightarrow{\epsilon\mapsto 0} & E & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & k & \xrightarrow{\cdot e} & k[\epsilon] & \longrightarrow & k & \longrightarrow & 0
\end{array}$$

we get $ob_{e_1}(a' \oplus t) = ob_\epsilon(t) = \theta(t) = *$, hence the result. ∎

Let G be a *Gdt* functor, $C \to A \in Psmex$; denote again by $\cdot$ the action of t_G on the fibres of $G(C) \to G(A)$. For $c \in G(C)$ let

$$Stab_G(c, C/A) = \{t \in t_G | t \cdot c = c\}.$$

PROPOSITION 4.12. *Let $\nu\colon F \to G$ be a morphism in Gdt. Then O_ν is complete if and only if, for every $C \overset{\alpha}{\to} A$ in Psmex, and every $c \in F(C)$ one has*

$$Stab_G(\nu(c), C/A) \subset \nu(t_F) \subset t_G.$$

Proof. The necessity of the condition follows from Proposition 4.2. To prove it's sufficient, assume we are in Set-up 4.10. As $F \in Gdt$, there exists

$d \in F(E)$ such that $\beta(d) = c$, $\gamma(d) = b$. Therefore $\gamma(\nu(d)) = \gamma(a)$, and there exist $t \in t_G$ such that $t \cdot \nu(d) = a$. By applying β we get $t \cdot \beta(\nu(d)) = t \cdot \nu(c) = \beta(a) = \nu(c)$; hence by assumption $t \in \nu(t_F)$. The result follows as in the proof of Proposition 4.11. ∎

COROLLARY 4.13. *Let ν: $F \to G$ be a morphism in Gdt. If either ν: $t_F \to t_G$ is surjective or G satisfies* (H4), *then O_ν is complete.*

Proof. In the first case $t_G/t_F = 0$, and in the second $Stab_G(c, C/A) = 0$ for all $C \to A$ and all $c \in G(c)$. ∎

For an example of a morphism ν in *Gdt* with O_ν non-complete, see Example 7.14.

5. PROREPRESENTABLE FUNCTORS

From now on we restrict our attention to *Gdt* functors. In this section we will study in particular prorepresentable functors, that is, functors isomorphic to h_R for some k-algebra R in $\widehat{Art}_k$. This is important as factorization Theorem 6.2 will allow us to reduce to such functors in many cases.

DEFINITION 5.1. Let $R \in \widehat{Art}_k$ and let $\mathbf{m} = \mathbf{m}_R$. Define $T^1_R = (\mathbf{m}/\mathbf{m}^2)^\vee$, $T^2_R = Ex(R, k)$. We call $\dim_k T^1_R$ the *embedding dimension* of R. For a small extension e

$$0 \to M \to B \to A \to 0,$$

define δ_e: $h_R(A) \times M^\vee \to T^2_R$ by $\delta_e(a, f) = a^* f_* e$.

LEMMA 5.2. (i) (T^2_R, δ_e) *is a complete linear obstruction theory for h_R.*

(ii) *Assume $R = P/I$ where P is a free power series algebra and $I \subset \mathbf{m}^2_P$. Then the small extension*

$$0 \to I/\mathbf{m}_P I \to P/\mathbf{m}_P I \to R \to 0$$

is universal, that is, any other small extension can be obtained from this by a unique pushforward. In particular T^2_R is canonically isomorphic to $(I/\mathbf{m}_P I)^\vee$.

Proof. (i) This is an exercise. (ii) Let $A \in Art_k$, $\phi \in h_R(A)$ and view ϕ as a morphism from R to A. Let e be an extension with A as target. ϕ induces a morphism $P \to A$; choose any lifting ψ to a morphism $P \to S(e)$. Such a lifting always exists, because to give a local morphism from $P = k[[x_1, \dots, x_n]]$ to $S \in \widehat{Art}_k$ is equivalent to choosing $f_1, \dots, f_n \in \mathbf{m}_S$ and requiring that $x_i \mapsto f_i$. As e is a small extension, ψ factors via $P/\mathbf{m}_P I$, inducing a linear map λ_ϕ: $I/\mathbf{m}_P I \to K(e)$. It is easy to verify that, as

$I \subset \mathbf{m}_P^2$, λ_ϕ does not depend on the lifting chosen. The condition that ϕ lifts to a $\tilde{\phi} \in h_R(S(e))$ is equivalent to saying that there is a lifting ψ such that $\psi(I) = 0$, that is, such that $\lambda_\phi = 0$. The result follows immediately. ∎

PROPOSITION 5.3. (i) *h_R is left-exact, in particular satisfies* (H1), ..., (H4);

(ii) *t_{h_R} is canonically isomorphic to T_R^1;*

(iii) *(T_R^2, δ_e) is the universal obstruction theory of h_R, and it is linear.*

Proof. Parts (i) and (ii) are classical results of Schlessinger [Sch]. To prove (iii), it is sufficient to find a morphism of obstruction theories α: $T_R^2 \to O_{h_R}$ such that its composition with the map $O_{h_R} \to T_R^2$ is the identity on T_R^2.

Consider an extension $e \in Ex(R, k) = T_R^2$

$$e: 0 \to k \to S \xrightarrow{\pi} R \to 0$$

and let $n \gg 0$ be an integer such that $m^n \cap I \subset mI$.

We have a morphism of extensions

$$\begin{array}{lccccccccc} \xi: & 0 \longrightarrow & I/\mathbf{m}I & \longrightarrow & P/\mathbf{m}I & \longrightarrow & R & \longrightarrow 0 \\ & & \| & & \downarrow & & \downarrow \pi_n & \\ \xi_n & 0 \longrightarrow & I/\mathbf{m}I & \longrightarrow & P/(\mathbf{m}I + \mathbf{m}^n) & \longrightarrow & R/\mathbf{m}^n & \longrightarrow 0 \end{array}$$

If $e = f_* \xi$, $f \in T_R^2$, we define $\alpha(f) = ob_{\xi_n}(\pi_n, f)$.

We leave to the reader the easy verification that α is a well defined morphism of obstruction theories and that it is an isomorphism. ∎

The following lemma is elementary but technically useful; we include a proof for lack of a suitable reference. The name comes from Milnor [Mi], where a similar result is proven in the complex, convergent set-up using differential topology.

LEMMA 5.4 (Curve Selection Lemma). *Let k be an algebraically closed field. Let $R \in \widehat{Art}_k$, and $g \in \mathbf{m}_R$ a non-nilpotent element. Then there is a local homomorphism ψ: $R \to k[[t]]$ such that $\psi(g) \neq 0$.*

Proof. We may assume that R is a domain (if not, quotient by any prime ideal not containing g). Let $d + 1 = \dim R$. If $d \geq 1$, then by Krull's Hauptidealsatz there exist $f_1, \ldots, f_d$ in R that generate an $\mathbf{m}$-primary ideal in $R/(g)$. Consider now the ring $R' = R/(f_1, \ldots, f_d)$. By Krull, it has dimension at least 1; however, $R'/(g)$ has dimension zero, hence $\dim R' \leq 1$. On the other hand g is not nilpotent in R', otherwise $\dim R' = \dim R'/(g)$. Hence we may replace R by R', and again assume that it is a domain. Now R has dimension 1 and is a domain; embed it into its integral closure $\tilde{R}$. As k is algebraically closed, $\tilde{R}$ is isomorphic to $k[[t]]$

(see for instance [Mat, Sect. 29]) and the inclusion $R \to \tilde{R}$ gives the required map ψ. ∎

The connection between the curve selection lemma and curvilinear obstructions is given by the following lemma, which we will often use without mentioning it in the sequel.

LEMMA 5.5. *Let $R \in \widehat{Art}_k$, write $R = P/I$ with P a power series algebra, and $I \subset \mathbf{m}_P^2$. Then $v \in T_R^2$ is a curvilinear obstruction if and only if there exists a morphism φ: $P \to k[[t]]$ in $\widehat{Art}_k$ with $\varphi(I)^{ext} = (t^p)$, $\varphi(f) = v(f)t^p$* mod t^{p+1} *for all $f \in I$.*

Proof. One implication is trivial: such a φ induces a morphism $R \to k[[t]]/(t^p)$, that is, an element of $h_R(k[[t]]/(t^p))$, and it is immediate to verify that the obstruction to lifting φ to $k[[t]]/(t^{p+1})$ is v.

Conversely, assume given ψ: $R \to k[[t]]/(t^p)$, such that $ob_e(\psi) = v$, where e is the extension

$$0 \longrightarrow k \xrightarrow{\cdot at^p} k[[t]]/(t^{p+1}) \longrightarrow k[[t]]/(t^p) \longrightarrow 0,$$

for some $a \in k$. Then the induced map $P \to k[[t]]/(t^p)$ can be extended to $P \to k[[t]]$ as in the proof of Lemma 5.2. ∎

LEMMA 5.6. *Let $R \in \widehat{Art}_k$. Then the following are equivalent*:

(i) *the functor h_R is smooth* (*hence R is a power series algebra*);

(ii) *h_R has no curvilinear obstructions*;

(iii) *there exists $N_0 \in \mathbb{N}$ such that the map $h_R(k[t]/t^{N+1}) \to h_R(k[t]/t^N)$ is surjective for $N \geq N_0$.*

Proof. The only nontrivial implication is (iii) ⇒ (i). Write $R = P/I$, with $P = k[[x_1, \dots, x_n]]$ and $I \subset \mathbf{m}_P^2$. Assume $I \neq 0$, that is, (i) does not hold. Let $f_1, \dots, f_r$ be a basis of $I/\mathbf{m}_P I$. Let $C \subset \mathbb{N}^n$ be the set of multi-indexes $J = (j_1, \dots, j_n)$ such that the monomial x^J appears with a nonzero coefficient in f_i for some $i = 1, \dots, r$. By assumption C is contained in $\{J | \Sigma j_i \geq 2\}$. It is an easy exercise in convex geometry to prove that there exist rational positive numbers $a_1, \dots, a_r, b$ such that $C \subset \{J | \Sigma a_i j_i \geq b\}$ and $C \cap \{J | \Sigma a_i j_i = b\} = \{J_0\}$. Choose $N \in \mathbb{N}$ such that $A_i = a_i N$ and $B = bN$ are all integers, and $B \geq N_0$. Define ψ: $P \to k[[t]]$ by requiring $\psi(x_i) = t^{A_i}$. We have $\psi(x^{J_0}) = t^B$, and $\psi(x^J) \in (t^{B+1})$ for all $J \in C$, $J \neq J_0$. Hence $\psi(I) \subset (t^B)$ (because $f_1, \dots, f_r$ generate I). Therefore ψ induces an element $\phi \in h_R(k[[t]]/t^B)$. As $\psi(I)$ is not contained in (t^{B+1}), ϕ does not lift to $h_R(k[[t]]/t^{B+1})$, contradicting (iii). ∎

EXAMPLE 5.7. (i) Curvilinear obstructions to h_R do not necessarily generate T_R^2 as a k-vector space. It is enough to consider $R =$

$k[x, y]/(x^3, y^3, x^2y^2)$. Then the class of x^2y^2 in $(T^2_R)^\vee$ is in the kernel of every curvilinear obstruction.

(ii) Let $R = k[x, y](x^3, y^3, x^2y^2)$, and let $S = k[x, y]/(x^3, y^3)$. Then the morphism $h_R \to h_S$ has no relative curvilinear obstructions.

(iii) Let $k = \mathbb{R}$, $R = k[[x, y]]/(x^2 + y^2)$, $S = k[[x, y]]/(x^2, y^2)$, and $R \to S$ be the natural projection. Then the morphism $h_S \to h_R$ has no curvilinear obstructions.

Remark. The above examples show that a morphism in *Gdt* without relative curvilinear obstructions is not necessarily smooth, even in the prorepresentable case and algebraically closed ground field. Therefore the above Lemma 5.6 does not generalize to the relative case; the best result we can prove is the following:

PROPOSITION 5.8. *Let k be an algebraically closed field, and let $S \to R$ be a morphism in $\widehat{Art}_k$. Assume that $h_R \to h_S$ has no relative curvilinear obstructions. Then $S_{red} \to R_{red}$ is smooth, and* $\dim R - \dim S = \dim t_R - \dim t_S$.

Proof. By Corollary 4.13 we know that O_ν is complete. So by Theorem 4.6 the map $t_R \to t_S$ is surjective. Hence we may assume that $R = P/I$ and $S = P'/I'$, with $P = k[[x_1, \dots, x_n]]$, $P' = k[[x_1, \dots, x_s]]$, $I' \subset I \subset \mathbf{m}_P^2$, $s \le n$.

Let $J \subset P$ be the ideal generated by $\sqrt{I'} \subset P'$. We need to prove that $J = \sqrt{I}$, as it is easy to see that this implies both statements; it is clearly enough to prove $\sqrt{I} \subset J$.

As a first step, we prove that $J = \sqrt{J}$. If $n = s$ there is nothing to prove, so by induction on $n - s$ we may assume that $s = n - 1$. Let $S_{red} = P'/\sqrt{I'}$, and let ϕ: $P \to S_{red}[[x_n]]$ be the natural surjection. As S_{red} is reduced, $\sqrt{\ker\phi} = \ker\phi$. Let $f \in \ker\phi$, and write $f = \sum f_i x_n^i$, with $f_i \in \sqrt{I'}$. Let $h_1, \dots, h_b$ be generators of $\sqrt{I'}$. Then $f_i = \sum f_{ia} h_a$, hence $f = \sum (x_n^i f_{ia}) h_a \in J$. Hence $\sqrt{I'} \subset \ker\phi \subset J$, which implies $\sqrt{J} \subset J$ and the claim.

It is therefore enough to prove $I \subset J$. Otherwise, by Lemma 5.4 there exists a ψ: $P \to k[[t]]$ in $\widehat{Art}_k$ such that $\psi(J) = 0$, $\psi(I) \ne 0$. Hence ψ defines an element $\alpha \in h_R(k[[t]]/(t^p))$ for some p such that $\nu(\alpha)$ can be lifted to $k[[t]]/(t^{p+1})$ but α can't. As O_ν is complete, there must be a nontrivial curvilinear obstruction. ∎

Note also that if k is not algebraically closed then Proposition 5.8 may fail, cf. Example 5.7(iii).

6. THE FACTORIZATION THEOREM WITH APPLICATIONS

In this section we prove the main application of the obstruction theory we have developed so far. Using factorization Theorem 6.2, a powerful generalization of Schlessinger's criterion for the existence of a hull, we can study *Gdt* functors without the (H3) condition. We also prove that for a *Gdt* functor F condition (L) is equivalent to the existence of a complete linear obstruction theory V; when this is the case, O_F is also linear in a unique way, and $O_F \to V$ is a linear monomorphism.

LEMMA 6.1 (Standard Smoothness Criterion). *Let ν: $F \to G$ be a morphism in Gdt. Then ν is smooth if and only if $t_F \to t_G \to * \to O_F \to O_G$ is an exact sequence of pointed sets.*

Proof. If ν is smooth, clearly $t_F \to t_G$ is surjective. Let $0 \to k \to B \to A \to 0$ be a small extension e, $a \in F(A)$. Then $ob_e(a) \in O_F$ maps to $ob_e(\nu(a))$ in O_G. If the latter is $*$, then $\nu(a)$ lifts to a $b' \in G(B)$ as O_G is complete; as ν is smooth, the pair (a, b') lifts to a $b \in F(B)$, hence $ob_e(a) = *$.

Assume now the sequence is exact. Let e be as before, and let $(a, b') \in \tilde{\nu}(e)$; let $a' \in G(A)$ be the common image of a and b'. Then $ob_e(a') = *$, as a' lifts to $G(B)$, hence $ob_e(a) = *$ by exactness. Therefore a lifts to some $b \in F(B)$. In general b does not map to b'; however, by Lemma 2.12(3) it maps to some $b'' \in G(B)$ which differs from b' by the action of an element $v \in t_G$ (v need not be unique). As $t_F \to t_G$ is surjective, v lifts to a $w \in t_F$; acting with w on b produces a lifting of a which maps to b', as required. ∎

Remark. The implication ν smooth $\Leftrightarrow$ sequence exact could also be proven using Theorem 4.6 and Corollary 4.13.

THEOREM 6.2 (Factorization Theorem). *Let ν: $F \to G$ be a morphism in Gdt, with F prorepresentable.*

Then there exists a factorization of ν, $F \xrightarrow{\mu} H \xrightarrow{\xi} G$ with the following properties:

(a) *H is prorepresentable, $t_F \to t_H$ is surjective, and $t_H \to t_G$ is injective.*

(b) $* \to O_H \xrightarrow{\xi} O_G$ *is exact in* Set_*.

If in addition G satisfies condition (H4), *then ξ is injective.*

Remark. According to the standard smoothness criterion ξ is a hull if and only if ν is surjective on tangent spaces.

Proof. Assume $F = h_R$ and let $m = \mathbf{m}_R$ be the maximal ideal of R. Define $R_q = R/m^q$. $\pi_q\colon R \to R_q$ the natural projection and $\nu_q = \nu(\pi_q) \in G(R_q)$. We now define by recursion a sequence $S_n, \xi_n \in G(S_n)$ together with morphisms $\mu_n\colon S_n \to R_n$ such that $\mu_n(\xi_n) = \nu_n$ and such that the diagrams

$$\begin{array}{ccc} S_n & \longrightarrow & S_{n-1} \\ \downarrow{\scriptstyle \mu_n} & & \downarrow{\scriptstyle \mu_{n-1}} \\ R_n & \longrightarrow & R_{n-1} \end{array}$$

commute. Let U be the image of the linear map $\nu_2\colon \operatorname{Hom}(R_2, k[\epsilon]) \to t_G$ and take $S_2 = k[U^\vee]$, $\mu_2\colon S \to R_2$ the transpose of ν_2 and ξ_2 the unique lifting of ν_2 to $G(S_2)$.

Assume $S_2, \dots, S_n$, $\xi_2, \dots, \xi_n$, and $\mu_2, \dots, \mu_n$ have already been defined. Set $V_{n+1} = R_{n+1} \times_{R_n} S_n$; then there exists $v_{n+1} \in G(V_{n+1})$ that lifts $\nu_{n+1} \in G(R_{n+1})$, $\xi_n \in G(S_n)$. Note that the embedding dimension of V_{n+1} is equal to that of S_n. Let $S_{n+1} \to V_{n+1}$ be the small extension (with fixed embedding dimension) which is maximal for the property of lifting v_{n+1} (cf. Lemma 4.8) and let ξ_{n+1} be a lifting. This defines S_n, ξ_n, μ_n. Note that $\mathbf{m}^n_{S_n} = 0$.

Take S, ξ, μ to be the inverse limits of S_n, ξ_n, μ_n and let $H = h_S$; we get a factorization of ν satisfying condition (a). It remains to prove that $* \to O_H \to O_G$ is exact.

Let $t = ob_e(f, 1) = f^*e \in Ex(S, k) = O_H$ be an obstruction where e is the small extension

$$0 \to k \to B \to A \to 0$$

and $f\colon S \to S_n \xrightarrow{f_n} A$, for some $n \gg 0$. Assume $\xi(t) = *$, hence that $\xi(f) = \xi(f_n)$ lifts to $G(B)$; by (H1), ξ_n lifts to $B(B \times_A S_n)$ and v_{n+1} lifts to $G(B \times_A V_{n+1})$. Therefore by the construction of S_{n+1} there exists a commuting homomorphism $S_{n+1} \to B \times_A V_{n+1} \to B$ and then f^*e is trivial in $Ex(S_{n+1}, k)$.

Assume now G satisfies (H4), we shall prove by induction on the length of $A \in Art_k$ that $\xi\colon H(A) \to G(A)$ is injective. Let $a, b \in H(A)$ be elements such that $\xi(a) = \xi(b)$ and let $p\colon A \to B$ be a principal small extension; by induction $p(a) = p(b)$ and then there exists a tangent vector $t \in t_H$ such that $t \cdot a = b$ and $\xi(t) \cdot \xi(a) = \xi(b)$. As G satisfies (H4), $\xi(t)$ must be 0 and then $t = 0$, $a = b$. ∎

Theorem 6.2 is based on Schlessinger's original proof of Corollary 6.3; hence we shall call a factorization as in Theorem 6.2 a "Schlessinger factorization."

COROLLARY 6.3. (i) (*Schlessinger*) *A functor with good deformation theory has a hull if and only if its tangent space is finite dimensional* (*that is*, (H3) *holds*).

(ii) *A Gdt functor F is left-exact if and only if it satisfies* (H4).

Proof. (i) One implication is trivial. Conversely, condition (H3) implies that there exists an $A \in Art_k$ and a morphism $h_A \to F$ inducing a surjection on tangent spaces (take $A = k \oplus t_F^{\vee}$). The result then follows from Lemma 6.1 and Theorem 6.2.

(ii) Assume F with property (H4). Let $p\colon A \to C$, $q\colon B \to C$ be morphisms in Art_k and consider the map $\eta\colon F(A \times_C B) \to F(A) \times_{F(C)} F(B)$.

(a) η is surjective. Take $(a, b) \in F(A) \times_{F(C)} F(B)$, according to Theorem 6.2 there exists an injective morphism $\xi\colon H \to F$ with H prorepresentable and $a' \in H(A), b' \in H(B)$ such that $\xi(a') = a$, $\xi(b') = b$. As ξ is injective $p(a') = q(b')$ and therefore, as H is left-exact we can lift (a', b') to $H(A \times_C B)$.

(b) η is injective. Denote $D = A \times_C B$ and let $r\colon D \to A$, $s\colon D \to B$ be the natural projections. Take $a, b \in F(D)$ such that $\eta(a) = \eta(b)$; as above there exists an injective morphism $\xi\colon H \to F$ with H left-exact and $a', b' \in H(D)$ such that $\xi(a') = a$, $\xi(b') = b$. By injectivity of ξ we have $r(a') = r(b')$, $s(a') = s(b')$, and therefore $a' = b'$. ∎

COROLLARY 6.4. *Let G be a Gdt functor. Then the following are equivalent*:

(i) *G is smooth*;

(ii) *G has no curvilinear obstructions*;

(iii) *there exists $N_0 \in \mathbb{N}$ such that the map $G(k[t]/t^{N+1}) \to G(k[t]/t^N)$ is surjective for $N \geq N_0$.*

Proof. If G is prorepresentable, this is Lemma 5.6. The only nontrivial implication is again (iii) $\Rightarrow$ (i). Assume G general and obstructed. Then there exists a small extension

$$e\colon 0 \to k \to B \to A \to 0$$

and an element $a \in G(A)$ such that $t = ob_e(a, 1) \neq *$. We now apply the factorization theorem to the morphism $a\colon h_A \to G$ and we get $h_A \to H \to G$ with H prorepresentable. Clearly t belongs to the image of O_H; so $O_H \neq *$, hence by Lemma 5.6 there exist nontrivial curvilinear obstructions in O_H of arbitrary high degree. The result now follows by property (b) of Theorem 6.2. ∎

LEMMA 6.5. *Let ν: $F \to G$ be a morphism of Gdt functors. Assume that O_F is linear and that V is a linear obstruction theory for G. Then the map ϕ: $O_F \to V$ induced by $O_F \to O_G$ is linear.*

Proof. Let $v, w \in O_F$. By Lemma 3.6 we can assume that $v = ob_e(a, f)$, $w = ob_e(a, g)$ for some small extension e and $a \in F(T(e))$, $f, g \in K(e)^\vee$. Then $ob_e(\nu(a), \cdot) = \phi \circ ob_e(a, \cdot)$: $K(e)^\vee \to V$. Hence

$$\phi(\lambda v + \mu w) = ob_e(\nu(a), \lambda f + \mu g) = \lambda ob_e(\nu(a), f)$$
$$+ \mu ob_e(\nu(a), g) = \lambda\phi(v) + \lambda\phi(w). \qquad \blacksquare$$

THEOREM 6.6. *Assume that the Gdt functor F has a complete linear obstruction theory (V, v_e). Then there exists a unique structure of vector space on O_F such that the universal obstruction theory is linear. Moreover the induced map ϕ: $O_F \to V$ is a linear monomorphism of vector spaces.*

Proof. It is enough to prove that $O_F \to V$ is injective. Take $v_1 \neq v_2$ in O_F. By Lemma 3.6, assume that $v_i = ob_e(a, f_i)$ for some small extension e, $a \in F(T(e))$, $f_1, f_2 \in K(e)^\vee$. a defines a morphism $h_{T(e)} \to F$. Apply Schlessinger's factorization Theorem 6.2 to get $h_{T(e)} \to H \to F$ with H prorepresentable, $* \to O_H \to O_F$ exact; v_1, v_2 belong to the image of O_H by functoriality. Then also $* \to O_H \to V$ is exact (as V is complete); on the other hand $O_H \to V$ is linear by Lemma 6.5, as O_H is linear by Proposition 5.3. Therefore $O_H \to V$ is injective, hence v_1 and v_2 have different images in V. $\blacksquare$

EXAMPLE 6.7. Let ν: $h_R \to F$ be a hull and let V be a linear complete obstruction space for F. Then the map ν induces a linear embedding $T_R^2 \subset V$.

EXAMPLE 6.8. A *Gdt* functor does not necessarily have a complete linear obstruction theory. Consider the Artinian k-algebra $R = k[[x, y]]/(x^2, xy, y^3)$ and, for every $\alpha \in k$, let g_α be the automorphism of R defined as $g_\alpha x = x + \alpha y^2$, $g_\alpha y = y$. Note that for $\alpha \neq 0$ the induced map g_α^*: $T_R^2 \to T_R^2$ is not trivial.

Define a functor F: $Art \to Set_*$ as $F(A) = h_R(A)/\sim$ where ϕ, ψ: $R \to A$, $\phi \sim \psi$ if and only if there exists α such that $\phi = \psi \circ g_\alpha$; an easy verification shows that the projection $h_R \xrightarrow{p} F$ is a hull, in particular it is smooth and the induced map $T_R^2 \xrightarrow{p} O_F$ has trivial kernel.

Assume O_F linear; then p is injective, but this is impossible since for every $\alpha \in k$, $p = p \circ g_\alpha^*$.

PROPOSITION 6.9. *Let R be a local noetherian complete k-algebra and let p: $h_R \to F$ be a smooth morphism of functors of Artin rings.*

Then O_F *is linear if and only if for every pair* $\phi, \psi \in h_R(A)$ *such that* $p(\phi) = p(\psi) = a$ *the induced linear maps* $\phi^*, \psi^*\colon Ex(A,k) \to Ex(R,k)$ *are equal.*

Proof. Since p is smooth the induced map $p\colon T^2_R \to O_F$ is surjective and has trivial kernel and in the above notation we have the equalities $O(p)\circ \phi^* = O(p)\circ \psi^* = O(a)\colon T^2_A = Ex(A,k) \to O_F$.

If O_F is linear than $O(p)$ is an isomorphism and then $\phi^* = \psi^*$.

Conversely assume $\phi^* = \psi^*$ whenever $p(\phi) = p(\psi)$; then we can define a complete linear obstruction theory of F by taking $Ex(R,k)$ as obstruction space and for obstruction maps $v_e(a,f) = f_* \phi^* e$ where $p(\phi) = a$. ∎

Remark. Let F be a *Gdt* functor. Given $A \in Art_k$, $a \in F(A)$, the set $Z(A,a) \subset Ex(A,k)$ was defined to be the set of extensions e such that a lifts to $F(S(e))$. By Proposition 4.3, $Z(A,a)$ is a vector subspace of $Ex(A,k)$; let $H(A,a)$ be the quotient vector space. Given a morphism $f\colon B \to A$ in Art_k and an element $b \in F(B)$, by Proposition 4.3, f^* induces a linear map (also denoted f^*) from $H(A, f(b))$ to $H(B,b)$.

LEMMA 6.10. *Let F be a Gdt functor, and assume F satisfies condition* (L). *Let* $f, g\colon B \to A$ *be morphisms in* Art_k, *and assume that* $f(b) = g(b) = a \in F(A)$. *Then* $f^* = g^*\colon H(A,a) \to H(B,b)$.

Proof. Let

$$e\colon 0 \to k \to C \to A \to 0$$

be an element of $Ex(A,k)$. We want to prove that $f^*e - g^*e \in Z(B,b)$. In the notation of Definition 2.9, $f^*e - g^*e = (f,g)^*\nabla(e)$ (cf. Lemma 4.9), where $(f,g)\colon B \to A \times_k A$. As $(f,g)(b) \in q^{-1}(\Delta_{F(A)})$, condition (L) implies that $(f,g)(b)$ lift to $\nabla(e)$; (H1) yields that b lifts to the small extension $f^*e - g^*e \in Ex(B,b)$. ∎

THEOREM 6.11. *Let F be a Gdt functor. Then* O_F *is linear if and only if condition* (L) *holds.*

Proof. One implication was proven in Lemma 4.9. Conversely, assume that (L) holds. By the alternative description of O_F after Theorem 3.2 and Lemma 4.8, O_F can be viewed as the disjoint union of the $H(A,a)$ modulo the equivalence relation generated by pullback.

The vector spaces $H(A,a)$ with the pullback morphisms are a subcategory of *Fvsp*; it is enough to prove that this subcategory is a *catégorie filtrante* (see Définition 2.7 and Proposition 2.8 in Exposé I of [SGA4]) hence its limit is a vector space and all structure maps are linear. We recall the axioms for a category I to be *filtrante*:

(PS1) Given morphisms $i \to j$ and $i \to j'$ in I there are morphisms $j \to k$ and $j' \to k$ such that the resulting diagram is commutative;

(PS2) Given two morphisms $i \to j$ there exist a morphism $j \to k$ such that the composed morphisms $i \to k$ coincide.

Moreover, it is required that I be nonempty and connected.

Lemma 6.10 says that (PS2) holds in the stronger sense that, given two objects, there is at most one morphism between them. In view of this, (PS1) is equivalent to saying that, given any two objects, there is a third to which they both map (the commutativity of the diagram is ensured by Corollary 6.12). Given A, B in Art_k and objects $a \in F(A)$, $b \in F(B)$, take $C = A \times_k B$; by (H1) there is $c \in F(C)$ mapping to $a \in F(A)$ and to $b \in F(B)$. ∎

COROLLARY 6.12. *Every functor F satisfying Schlessinger conditions* (H1), (H2), *and* (H4) *has a complete linear obstruction theory*.

Proof. This follows immediately from Lemma 2.11 and Corollary 6.3. ∎

We are now in the position to prove that, if k has characteristic 0, then the T^1-lifting theorem holds for every *Gdot* functor.

DEFINITION 6.13 [Ran2, Kaw]. Let $A_n = k[t]/(t^{n+1})$,

$$B_n = k[x, y]/\left(x^{n+1}, y^2\right),$$

and let $\beta_n \colon B_n \to A_n$ be the map defined by $x \mapsto t$, $y \mapsto 0$. A *Gdt* functor F has the T^1-*lifting property* if, for every $n \in \mathbb{N}$, the natural map

$$F(B_{n+1}) \to F(B_n) \times_{F(A_n)} F(A_{n+1})$$

is surjective. In [F-M] we have proved the following

THEOREM 6.14. *Let F be a functor with good deformation theory such that*:

(a) *F has a complete linear obstruction theory*.

(b) *F has the* T^1-*lifting property*.

If k has characteristic 0 *then for every* $n \geq 0$ *the map* $F(A_{n+1}) \to F(A_n)$ *is surjective*.

Proof. See [F-M], Theorem A. ∎

COROLLARY 6.15. *Let F be a functor with good deformation and obstruction theory. If k has characteristic* 0 *and F has the T^1-lifting property then F is smooth.*

Proof. This is an immediate consequence of Theorems 6.11 and 6.14 and Corollary 6.4. ∎

7. GROUP AND AUTOMORPHISM FUNCTORS

By a group functor of Artin rings we shall mean a functor G from Art_k to the category of groups such that the associated functor from Art_k to Set_* which forgets the group structure is a functor of Artin rings in the sense of Definition 2.1.

EXAMPLE 7.1. If R is the local ring at the unit element of a group scheme [Mum, Sect. 11], then $G = h_R$ is a group functor satisfying conditions (H1), (H2), and (H4).

EXAMPLE 7.2. Let X be a noetherian separated scheme over k, for every $A \in Art_k$ denote by $X_A = X \times_k Spec\, A$, and let i: $X \to X_A$, p: $X_A \to Spec\, A$ be respectively the closed immersion and the projection. We then define

$$Aut_X(A) = \{\phi: X_A \to X_A | p \circ \phi = p,\ \phi \circ i = i\}.$$

It is clear that Aut_X is a group functor and, according to [Sch, 3.11], satisfies conditions (H1), (H2), and (H4) with tangent space $t_{Aut_X} = H^0(X, \theta_X)$ where $\theta_X = \Omega_X^\vee$ is the sheaf of tangent vector fields. Note that if X is not proper over k in general the tangent space of Aut_X is not finite dimensional; indeed this is one of our motivations to develop a theory where condition (H3) plays a very marginal role.

It is not clear to us if every *Gdt* group functor satisfies also condition (H4); however, every *Gdt* group functor has a complete linear obstruction theory, as follows from the following

PROPOSITION 7.3. *Let F be a Gdt group functor. Then F satisfies condition* (L).

Proof. Let $B \to A$ be a small extension. Let

$$F(C) \overset{p}{\to} F(A \times_k A) \overset{q}{\to} F(A) \times F(A)$$

be as in Definition 2.9 and δ: $F(A) \to F(C)$ be a lifting of the diagonal map.

Let $a \in F(A \times_k A)$ such that $q(a) = (d, d)$. Then since $q(a \cdot (p\delta(d))^{-1}) = (*, *)$ we can assume without loss of generality that $q(a) = (*, *)$ and then, according to Lemma 7.4, a lifts to $F(B \times_k B)$ and hence also to $F(C)$. ∎

LEMMA 7.4. *Let F be a Gdt functor and for every pair $A, B \in Art_k$ let $K(A, B)$ be the kernel of the natural map $F(A \times_k B) \to F(A) \times F(B)$. If $A_1 \to A_2$, $B_1 \to B_2$ are surjective morphisms in Art_k, then the natural map $K(A_1, B_1) \to K(A_2, B_2)$ is surjective.*

Proof. It is clearly sufficient to prove the lemma when $B_1 = B_2 = B$ and $A_1 \to A_2$ is a principal small extension. In this case the result is an immediate consequence of condition (H1) applied to the cartesian diagram

$$\begin{array}{ccc} A_1 \times_k B & \longrightarrow & A_2 \times_k B \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_2. \end{array}$$

∎

Given a functor of Artin rings F and a group functor of Artin rings G, by a G-action on F we shall mean a morphism $G \times F \to F$ such that for every $A \in Art_k$

$$G(A) \times F(A) \to F(A) \qquad (g, a) \to a^g$$

is a $G(A)$-action on $F(A)$ in the usual sense. One can clearly define in the obvious way the quotient functor F/G.

We shall see below that in most concrete cases G will be a smooth *Gdt* group functor, in this case F/G inherits many properties of F, in particular we have the following

PROPOSITION 7.5. *Let F be a Gdt functor and G be a smooth Gdt group functor. Then F/G is a Gdt functor and the natural obstruction map $O_F \to O_{F/G}$ is bijective. In particular the projection $F \to F/G$ is smooth and if F satisfies condition* (L) *the same holds for F/G* (*note that the analogous statement for the condition* (H4) *is false*).

Proof. The proof that F/G is a *Gdt* functor is straightforward and it is left as an exercise (see also [Wa, 1.1.6]). Since $F \to F/G$ is surjective the natural map $O_F \to O_{F/G}$ is also surjective.

Consider now $a \in F(A)$, $e \in Ex(A, k)$, and $g \in G(A)$; according to Lemma 7.6 it is sufficient to prove that $ob_e(a) = ob_e(a^g)$.

Let π_i: $A \times_k A \to A$, $i = 1, 2$ be the projections, δ: $A \to A \times_k A$ be the diagonal, and, in the notation of Lemma 4.9,

$$\nabla e : 0 \to k \to C \to A \times_k A \to 0.$$

Let $c = \delta(a) \in F(A \times_k A)$ and let $\tau \in G(A \times_k A)$ such that $\pi_1(\tau) = 1$, $\pi_2(\tau) = g$, as δ lifts to a morphism $A \to C$ and G is smooth we have that c and τ lift to $F(C)$ and $G(C)$, respectively, and therefore also c^τ lifts to $F(C)$. Since $\pi_1(c^\tau) = \pi_1(c)^{\pi_1(\tau)} = a$, $\pi_2(c^\tau) = \pi_2(c)^{\pi_2(\tau)} = a^g$ we have by Lemma 4.9 $ob_e(a) = ob_e(a^g)$. ∎

LEMMA 7.6. *Let ψ: $F \to F'$ be a surjective morphism of functors of Artin rings with universal obstruction theories $(ob, O_F), (ob', O_{F'})$, respectively. The induced obstruction map $o(\psi)$: $O_F \to O_{F'}$ is bijective if and only if for every small extension*

$$e : 0 \to k \to B \to A \to 0$$

and for every $a_1, a_2 \in F(A)$ such that $\psi(a_1) = \psi(a_2)$ we have $ob_e(a_1) = ob_e(a_2) \in O_F$.

Proof. The condition of the lemma is clearly necessary; in order to show that it is also sufficient we define for every small extension

$$e : 0 \to M \to B \xrightarrow{\pi} A \to 0$$

a map v_e: $F'(A) \times M^\vee \to O_F$ by setting $v_e(\psi(a), f) = ob_e(a, f)$. It is clear that v_e is well defined and (v_e, O_F) is an obstruction theory of F' such that $ob'_e = o(\psi) \circ v_e$. By universality there exists β: $O_{F'} \to O_F$ such that $v_e = \beta \circ ob'_e$ and β is exactly the inverse of $o(\psi)$. ∎

EXAMPLE 7.7. Let $X = \mathbb{A}^n_k$ be the affine space over k. Then there exists a natural bijection between $Aut_X(A)$ (Example 7.2) and the set of k-algebra homomorphisms ϕ: $k[x_1, \ldots, x_n] \to A[x_1, \ldots, x_n]$ such that $\pi \circ \phi = 1$, where π: $A[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ is the projection.

In this case it is immediate to observe that Aut_X is smooth. This is in fact a special case of Example 2.4, where $V = P^n$, and $P = k[x_1, \ldots, x_n]$ as a k-vector space.

EXAMPLE 7.8. Let $X \subset \mathbb{A}^n_k$ be an affine scheme of finite type over k, F the functor of embedded deformations of X in $\mathbb{A}^n_k$, and G the automorphism functor of the affine space (Example 7.7). Then G acts on F and the quotient functor F/G is the functor Def_X of infinitesimal deformations of X [Ar1]. It is well known (and easy to see) that F satisfies conditions (H1), (H2), and (H4). In particular by Proposition 7.5 it follows that Def_X is a Gdt functor with a complete linear obstruction theory.

PROPOSITION 7.9. *Let X be a noetherian separated scheme over k. If k has characteristic* 0, *then Aut_X is smooth.*

Proof. This is a special case of Theorem 7.19 below. ∎

Remark. In the affine case Proposition 7.9 was proved in [Wa, 1.3.1] by using (formal) integration of vector fields.

EXAMPLE 7.10. If the characteristic of k is a positive integer p then in general Aut_X is not smooth. Consider for example the affine scheme $X = Spec(R)$, $R = k[x]/(x^p)$ and let $g \in Aut_X(A_{p-1})$ be defined by g: $k[x,t]/(x^p, t^p) \to k[x,t]/(x^p, t^p)$, $g(t) = t$, $g(x) = x + t$. This g does not lift to $Aut_X(A_p)$; in fact every lifting g': $k[x,t]/(x^p, t^{p+1}) \to k[x,t]/(x^p, t^{p+1})$ should have the form $g'(t) = t$, $g'(x) = x + t + at^p$ and then $g'(x^p) = t^p \neq 0$.

LEMMA 7.11. *Let X be a scheme as in Proposition* 7.9, *let*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

be a cartesian diagram in Art_k with $C \to D$ a small extension, and let $X_A \to Spec(A)$ be a deformation of X inducing trivial deformations over $Spec(B)$ and $Spec(C)$. If Aut_X is smooth then X_A is a trivial deformation.

Proof. Let X_B, X_C, X_D be the induced deformations over $Spec(B)$, $Spec(C)$, $Spec(D)$. By assumption there exist isomorphisms f: $X \times_k Spec(B) \to X_B$, g: $X \times_k Spec(C) \to X_C$, inducing isomorphisms $\bar{f}, \bar{g}$: $X \times_k Spec(D) \to X_D$. Let $\tau \in Aut_X(C)$ be a lifting of $\bar{f}\bar{g}^{-1}$. Then, considering possibly the composition of g with τ, we can assume $\bar{f} = \bar{g}$ and then it is possible to glue the maps f, g to an isomorphism $X \times_k Spec(A) \to X_A$. ∎

COROLLARY 7.12. *Let X be a scheme as in Proposition* 7.9, *F a Gdt functor, and μ: $F \to Def_X$ a morphism of functors. If the characteristic of k is* 0 *then $K = \ker \mu$ is a Gdt functor.*

Proof. (cf. [G-K, Sect. 1]). Since $K \subset F$ it is enough to prove that K satisfies condition (H1).

Given a cartesian diagram in Art_k

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

with $C \to D$ small extension and $a \in F(A)$ such that $\mu\alpha(a) = *$, $\mu\beta(a) = *$ we know from Lemma 7.11 that also $\mu(a) = *$ and therefore the kernel of μ satisfies (H1). ∎

The proof of the above result suggests an example of hull with nontrivial kernel.

EXAMPLE 7.13. Let X be the affine scheme of Example 7.10 and let $h_R \overset{\mu}{\to} Def_X$ be a hull. Then the kernel of μ is nontrivial. In fact X is an affine hypersurface, hence, according to [Ar1, Sect. 4], $R = k[[z_0, \ldots, z_{p-1}]]$ is a power series ring with p generators and the image under μ of a morphism $f\colon R \to A$ is the isomorphism class of the deformation $Spec(A[x]/(x^p + \sum f(z_i)x^i))$. A nontrivial element of the kernel of μ is the pair A, f where $A = k[t]/(t^{n+1})$, $n > p$, and $f\colon R \to A$ is given by $f(z_0) = t^p$, $f(z_1) = \cdots = f(z_{p-1}) = 0$; the induced deformation $Spec(k[x,t]/(x^p + t^p, t^{n+1}))$ is isomorphic to $Spec(k[y,t]/(y^p, t^{n+1}))$, by $y = x + t$.

Note that if $\phi\colon F \to G$ is a morphism of Gdt functors and O_ϕ is complete then for every morphism $B \to A$ in Art_k and every small extension $A' \to A$ the kernel of $\eta\colon G(A' \times_A B) \to G(A') \times_{G(A)} G(B)$ is contained in the image of ϕ. In particular if $F = *$, O_ϕ complete, then the kernel of η is trivial and, by the same argument of Corollary 7.12 the kernel of every morphism of Gdt functors $H \to G$ is again a Gdt functor.

EXAMPLE 7.14. Let X be the scheme of Example 7.13 and let $\nu\colon * \to Def_X$ be the trivial morphism. Then O_ν is not complete.

In fact if O_ν were complete then the kernel of the hull $\mu\colon h_R \to Def_X$ would be a Gdt functor and therefore trivial since μ is an isomorphism on tangent spaces.

Let $F \in Fun$; define $K_F\colon Art_k \to Set_*$ by setting for every $A \in Art_k$

$$K_F(A) = \ker\Big(F(A \otimes_k k[\epsilon]) \overset{\pi}{\to} F(A)\Big),$$

where π is the map induced by the natural projection. Note that K_F is a covariant functor but in general not in Fun, because $K_F(k) = t_F$. The introduction of K_F is motivated by the following result.

Lemma 7.15. *Let G be a Gdt group functor. Then the T^1-lifting property holds for G if and only if the natural map $K_G(A_{n+1}) \to K_G(A_n)$ is surjective for every n.*

Proof. Note first that $\beta_n\colon B_n \to A_n$ has a splitting given by $t \mapsto x$. Note also that $B_n = A_n \otimes_k k[\epsilon]$, hence $K_G(A_n) = \ker(G(B_n) \to G(A_n))$. Choose an element (a', b) in $G(A_{n+1}) \times_{G(A_n)} G(B_n)$; we want to prove that (a', b) lifts to $G(B_{n+1})$. Using the section of β_{n+1} we can find $\bar{b}' \in G(B'_{n+1})$ mapping to a'. Let $\bar{b}$ be the image of $\bar{b}'$ in $G(B_n)$. As $\bar{b}$ and b have the same image in $G(A_n)$, there exists an element $g \in K_G(A_n)$ such that $b = g\bar{b}$. By assumption g can be lifted to $g' \in K_G(B_n)$; it is then easy to verify that $b' = g'\bar{b}'$ is the required lifting. ∎

Lemma 7.16. *If $H = h_R$ is a prorepresentable functor then $K_H(A) = t_H \otimes A$; in particular $K_H(B) \to K_H(A)$ is surjective whenever $B \to A$ is.*

Proof. For every $a \in R$ let $\bar{a} \in k$ be its class in the residue field. The elements of $K_H(A)$ are exactly the morphisms $R \to A \otimes_k k[\epsilon]$ of the form $a \to \bar{a} + \epsilon\phi(a)$ where $\phi \in \mathrm{Der}_k(R, A)$ and the R-module structure on A is induced by the projection $R \to k$. Hence $K_H(A) = \mathrm{Der}_k(R, A) = \mathrm{Hom}_k(\Omega_{R/k}, A) = t_H \otimes A$. ∎

Lemma 7.17. *Let $\xi\colon H \to F$ be a morphism of Gdt functors with H prorepresentable and let $t \in t_F$. Then there exists a factorization $H \to K \to F$ with K prorepresentable and t in the image of t_K.*

Proof. Let $H = h_R$ and assume that the morphism ξ is given by a coherent sequence $\xi_n \in F(R_n)$, where $R_n = R/\mathbf{m}^n$. As F is *Gdt* we have $F(R_2[\epsilon]) = F(R_2) \times t_F$ and therefore there exists $\eta_2 \in F(R_2[\epsilon])$ whose projections are exactly ξ_2 and t.

For every $n \geq 2$ we have a cartesian diagram in Art_k

$$\begin{array}{ccc} R_{n+1}[\epsilon] & \longrightarrow & R_n[\epsilon] \\ \downarrow & & \downarrow \\ R_{n+1} & \longrightarrow & R_n \end{array}$$

and therefore it is easy to see, using (H1), that there exists a coherent sequence $\eta_n \in F(R_n[\epsilon])$ which lifts ξ_n and η_2.

Now $\eta\colon K = h_{R[\epsilon]} \to F$ is the required factorization. ∎

Lemma 7.18. *Let F be a Gdt functor, $A \in Art_k$, and $a \in K_F(A)$. Then*

(i) *there exists a morphism $H \to F$ with H prorepresentable such that a lifts to $K_H(A)$;*

(ii) *for every small extension $B \to A$ the map $K_G(B) \to K_G(A)$ is onto.*

Proof. Consider the splitting of the projection $\pi\colon A \otimes_k k[\epsilon] \to A$ into a sequence of principal small extensions

$$A \otimes_k k[\epsilon] = C_n \to C_{n-1} \to \cdots \to C_0 = A$$

and denote by $\pi_i\colon C_i \to A$, $v_i\colon A \otimes_k k[\epsilon] \to C_i$ the partial projections; put $a_i = v_i(a)$, by assumption $\pi_i(a_i) = *$.

We now prove by induction the following result.

For every $i = 0, \dots, n$ there exists a prorepresentable functor H_i, a morphism $\phi_i\colon H_i \to F$, and a $b_i \in H_i(C_i)$ such that $\phi_i(b_i) = a_i$, $\pi_i(b_i) = *$.

For $i = 0$ the above result is trivially true. Assume there are given H_i, b_i, and ϕ_i as above for an index $i < n$. Since a_i lifts to $F(C_{i+1})$, according to factorization Theorem 6.2, we can assume without loss of generality that b_i lifts to some $s_{i+1} \in H_i(C_{i+1})$. As $C_{i+1} \to C_i$ is a principal small extension there exists a transitive action of t_F on the fibres of $F(C_{i+1}) \to F(C_i)$ and then there exists $t \in t_F$ such that $t \cdot \phi_i(s_{i+1}) = a_{i+1}$. By Lemma 7.17 there exists a factorization $H_i \xrightarrow{\alpha} H_{i+1} \xrightarrow{\phi_{i+1}} F$ such that t lifts to $t_{H_{i+1}}$ and therefore also the pair $(a_{i+1}, \alpha(b_i))$ lifts to $H_{i+1}(C_{i+1})$. This proves (i); to prove (ii) let $a \in K_F(A)$ be a fixed element and let $B \to A$ be a small extension in Art_k. Let $H \to F$ be a morphism with H prorepresentable such that a lifts to $a' \in K_H(A)$; by Lemma 7.16, a' lifts to $K_H(B)$ and therefore also a lifts to $K_F(B)$. ∎

THEOREM 7.19. *Let G be a Gdt group functor over a field k of characteristic* 0. *Then G is smooth.*

Proof. By Proposition 7.3 and Corollary 6.15 it is sufficient to show that G has the T^1-lifting property. By Lemma 7.15 this is equivalent to the surjectivity of $K_G(A_n) \to K_G(A_{n-1})$ for every $n \geq 2$; this is an immediate consequence of Lemma 7.18(ii). ∎

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